

# A necessary and sufficient condition for the subexponentiality of product distribution

Hui Xu Fengyang Cheng Yuebao Wang<sup>\*†</sup>

School of Mathematical Sciences, Soochow University, Suzhou 215006, China

## Abstract

Let  $X$  and  $Y$  be two independent non-negative random variables with corresponding distributions  $F$  and  $G$ . Denote the distribution of  $XY$  by  $H$  which is called "product distribution" for short. When  $F$  is subexponential, Cline and Samorodnitsky (1994) have proposed conditions for  $H$  to be subexponential too. Taking into account a result by Tang (2008) in the case that  $G$  is supported on  $[0, \infty)$ , we find a necessary and sufficient condition for  $H$  to be subexponential if  $F$  is. Further, we give an example to show that the condition (d) in Theorem 2.1 of the above paper is not necessary. Correspondingly, we also give two simple results on the closure property under the product distribution root, that is if  $H$  is subexponential, under certain conditions,  $F$  is the same one. Finally, as applications of the above results, we derive the asymptotic estimate of the finite time ruin probability in a discrete-time insurance risk model with random interest rate.

*Keywords:* necessary and sufficient condition; product distribution; closure property; subexponential distribution; ruin probability; asymptotic estimate

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## 1 Introduction and main results

Throughout the paper, we always assume that  $X$  and  $Y$  are two independent random variables with distributions  $F$  supported on  $[0, \infty)$ , that is,  $\overline{F}(x) = 1 - F(x) > 0$  for all  $x \geq 0$  and  $\overline{F}(x) = 1$  for all  $x < 0$ , and  $G$  supported on  $[0, s]$  for some constant  $s > 0$  or  $[0, \infty)$ , respectively. Then the distribution  $H$  of the random variable  $XY$  is called the product distribution for short. And the distribution  $F * G$  of the random variable  $X + Y$  is called their convolution.

It is well known that the product distribution plays an important role in many applied probability fields and has attracted a fair amount of interest in recent years, see, for example, Cline and Samorodnitsky (1994), Li and Tang (2015), Samorodnitsky and Sun (2016). Especially in finance and insurance, many important research objects, such as income (or discount), can be expressed as the product of initial capital and interest rate (or its function), see Section 4 of present paper for the details. In the research of product distribution, the closure of product distribution for certain distribution class is one of the important research topics. Precisely, if  $F$  belongs to a certain distribution class, under what condition, then the product distribution  $H$  also belongs to the same one? Compared with the closure of convolution, however, the study of the closure of

<sup>\*</sup>Research supported by National Science Foundation of China, Grant No.11071182

<sup>†</sup>Corresponding author. Telephone: 86 512 67422726. Fax: 86 512 65112637. E-mail: ybwang@suda.edu.cn

product distribution seems to be more difficult in some sense. Therefore, here a lot of interesting questions which have not been solved in a long period. This paper attempts to address some of these problems. To this end, we first introduce some concepts of related distribution classes.

In this paper, all limit relationships are for  $x \rightarrow \infty$  unless otherwise stated. For two positive functions  $f(\cdot)$  and  $g(\cdot)$ , we write  $f(x) \sim g(x)$  if  $\lim f(x)/g(x) = 1$ , write  $f(x) = o(g(x))$  if  $\lim f(x)/g(x) = 0$ , write  $f(x) = O(g(x))$  if  $\limsup f(x)/g(x) < \infty$ , and write  $f(x) \lesssim g(x)$  if  $\limsup f(x)/g(x) \leq 1$ . The indicator function  $\mathbf{1}(A)$  of an event  $A$  takes the value 1 if the event occurs and the value 0 otherwise.

A distribution  $V$  supported on  $(-\infty, \infty)$  belongs to the distribution class  $\mathcal{L}(\gamma)$  for some  $\gamma \geq 0$ , if for any fixed  $t > 0$ ,

$$\overline{V}(x-t) \sim e^{\gamma t} \overline{V}(x).$$

Here, when  $\gamma > 0$  and the distribution  $F$  is lattice, then  $x$  and  $t$  should be restricted to values of the lattice span. And when  $\gamma = 0$ ,  $\mathcal{L}(0)$  represents the well known long-tailed distribution class, denoted by  $\mathcal{L}$ .

A famous subclass of long-tailed class is introduced as follows. A distribution  $V$  supported on  $[0, \infty)$  belongs to the subexponential distribution class  $\mathcal{S}$ , denoted by  $V \in \mathcal{S}$ , if

$$\overline{V^{*2}}(x) \sim 2\overline{V}(x),$$

where  $V^{*2}$  denotes the two-fold convolution of  $V$  with itself. Further, say that a distribution  $V$  supported on  $(-\infty, \infty)$  belongs to subexponential distribution class, if  $V_+(x) = V(x)\mathbf{1}(x \geq 0)$  is subexponential.

The class  $\mathcal{S}$  were introduced by Chistyakov (1964) and have been applied to many fields of probability theory, such as risk model, queueing system, branching processes, and so on, see Embrechts et al. (1997), Asmussen (2000) and Foss et al. (2013) for the details. It is well known that if  $V \in \mathcal{L}$ , then it is heavy-tailed, that is  $\int_0^\infty e^{\alpha y} V(dy) = \infty$  for all  $\alpha > 0$ .

Another famous heavy-tailed class is the dominated varying-tailed distribution class  $\mathcal{D}$  introduced by Feller (1971). we say that a distribution  $V$  supported on  $(-\infty, \infty)$  belongs to class  $\mathcal{D}$  if

$$\overline{V}(tx) = O(\overline{V}(x))$$

holds for some (and hence for all)  $0 < t < 1$ .

An important subclass of dominated varying-tailed class and subexponential class is regularly varying-tailed distribution class denoted by  $\mathcal{R}$ . Recall that a distribution  $V$  is regularly varying-tailed, if for some  $\alpha \geq 0$  and all  $t > 1$ ,

$$\overline{V}(tx) \sim t^{-\alpha} \overline{V}(x).$$

Resnick (1987) and Bingham et al. (1987) systematically studied the regularly varying function which contains regularly varying-tailed distribution function.

Another subclass of subexponential class is denoted by  $\mathcal{A}$ , in which a distribution  $V$  is subexponential and the condition

$$\limsup \overline{V}(tx)/\overline{V}(x) < 1$$

holds for some  $t > 1$ .

On the study of the closure of product distribution, Embrechts and Goldie (1980) gave a result that, if  $F \in \mathcal{R}$  and  $\overline{G}(x) = o(\overline{F}(x))$ , then  $H \in \mathcal{R}$ . When  $F \in \mathcal{S}$ , Cline and Samorodnitsky (1994)

proved  $H \in \mathcal{S}$  under some sufficient conditions. Samorodnitsky and Sun (2016) extended the result in the sense of the product probability space. Under more relaxed condition, Tang (2006) investigated the closure of product distribution for class  $\mathcal{A}$ . In addition, Tang (2008) gave an necessary and sufficient condition for  $H \in \mathcal{L}$ , when  $F \in \mathcal{L}(\gamma)$  for some  $\gamma \geq 0$ .

Now, we recall Theorem 2.1 of Cline and Samorodnitsky (1994) and Theorem 1.1 of Tang (2008) in detail.

**Theorem 1.A.** *If  $F \in \mathcal{S}$  and there is a function  $a(\cdot) : [0, \infty) \rightarrow (0, \infty)$  satisfying the following, then  $H \in \mathcal{S}$ .*

- (a)  $a(x) \uparrow \infty$ ;
- (b)  $a(x)/x \downarrow 0$ ;
- (c)  $\overline{F}(x - a(x)) \sim \overline{F}(x)$ ;
- (d)  $\overline{G}(a(x)) = o(\overline{H}(x))$ .

Here, the condition (c) means that distribution  $F$  is  $a$ -insensitive in the asymptotic sense for the function  $a$ . In the following,  $D[V]$  represents the set of all positive discontinuities of a distribution  $V$ .

**Theorem 1.B.** *Assume that  $F \in \mathcal{L}(\gamma)$  for some  $\gamma \geq 0$  and  $G$  is supported on  $[0, \infty)$ , then  $H \in \mathcal{L}$  if and only if  $D[F] = \emptyset$ , or when  $D[F] \neq \emptyset$ , it holds that*

$$\overline{G}(x/d) - \overline{G}((x+1)/d) = o(\overline{H}(x)) \quad \text{for all } d \in D[F]. \quad (1.1)$$

This condition has aroused our interest. We will find that, for the case that  $F \in \mathcal{S}$ , the condition also plays a very important role in this paper.

The two important results on product distribution have produced some interesting problems naturally.

First of all, we want to know whether these four sufficient conditions (a)-(d) in Theorem 1.A are necessary for  $H \in \mathcal{S}$ ? Corollary 2.5 of Cline and Samorodnitsky (1994) once pointed out that, for a distribution  $F \in \mathcal{L}$ , there is a function  $a(\cdot) : [0, \infty) \rightarrow (0, \infty)$  satisfying the conditions (a)-(c). Thus, in the conditions (a)-(d), the condition (d) is a key for the theorem. Further, Lemma 3.2 of Tang (2006) noted that, the conditions (a), (b) and (d) are satisfied if and only if the relation

$$\overline{G}(x) = o(\overline{H}(bt)) \quad \text{for every } b > 0$$

holds. However, we find that, there are two distributions  $F$  and  $G$  such that  $F \in \mathcal{R} \subset \mathcal{S}$ ,  $G \notin \mathcal{L}$  and  $H \in \mathcal{S}$ , while the above relation does not hold. In other words, the condition (d), even the above relation, is not necessary for  $H \in \mathcal{S}$ . See Subsection 3.1 for details.

Further, we naturally want to know that, whether we can cancel condition (d) in the theorem? Then, whether we can find a necessary and sufficient condition for  $H \in \mathcal{S}$  when  $F \in \mathcal{S}$ ? More specifically, is it possible that conditions  $D[F] = \emptyset$  or (1.1) in Theorem 1.B can be used as the necessary and sufficient condition? In this regard, our answer is positive. In addition, if  $F \in \mathcal{S}$  and  $G$  is supported on  $[0, s]$  for some constant  $s > 0$ , then the conditions (a)-(d) in Theorem 1.A are automatically satisfied, thus  $H \in \mathcal{S}$ , see Corollary 2.5 of Cline and Samorodnitsky (1994). Therefore, we only need to study the case that  $G$  has a infinite support.

**Theorem 1.1.** *Assume that  $F \in \mathcal{S}$  and  $G$  is supported on  $[0, \infty)$ , then  $H \in \mathcal{S}$  if and only if  $D[F] = \emptyset$ , or when  $D[F] \neq \emptyset$ , the condition (1.1) holds.*

**Remark 1.1.** *i) Because  $H$  is often an unknown distribution, the condition (1.1) may be difficult to be verified. Therefore, Corollary 1.1 of Tang (2008) proposed three sufficient conditions (A), (B) and (C) on known distributions  $F$  and  $G$  for the condition. In addition, we are also willing to provide a condition slightly weaker than the condition (B) in the above corollary. That is if  $G^{*k} \in \mathcal{L}$  for some  $k \geq 1$ , then by Theorem 2.1(1a) of Xu et al. (2015), for any constant  $d > 0$ , we can get*

$$\overline{G}(x/d) - \overline{G}((x+1)/d) = o(\overline{G^{*k}}(x/d)).$$

*Further, from the following fact that*

$$\overline{G^{*k}}(x/d) \leq k\overline{G}(x/(kd)) \leq k\overline{H}(x)/\overline{F}(kd),$$

*the condition (1.1) holds. We note that, there exist some distributions  $G$  such that  $G \notin \mathcal{L}$ , while  $G^{*k} \in \mathcal{L}$  for some  $k \geq 2$ . Specifically, such distributions can be found in the proofs of Theorem 2.2 and Proposition 2.1 of Xu et al. (2015), respectively.*

*ii) The subexponentiality of  $H$ , however, does not imply  $F \in \mathcal{S}$  or  $G \in \mathcal{S}$ . That is, the condition that  $F$  (or  $G$ )  $\in \mathcal{S}$  is not necessary for  $H \in \mathcal{S}$ . In the Example 2.1 of Tang (2008), there are two standard exponential distributions  $F$  and  $G$  with parameters  $\lambda_F$  and  $\lambda_G$  respectively, while  $H$  still belongs to the subexponential class. In addition, let  $1 < \alpha < 2$  and*

$$\overline{F}(x) = \overline{G}(x) = \mathbf{1}(x \leq 0) + e^{-x^\alpha} \mathbf{1}(x > 0) \text{ for all } x.$$

*Clearly,  $F, G \notin \mathcal{L}(\gamma)$  for any  $\gamma \geq 0$ . Further, by Lemma 2.1 of Arendarczyk and Dębicki (2011), we can get*

$$\overline{H}(x) \sim \sqrt{\pi} x^{\alpha/4} e^{-2x^{\alpha/2}}.$$

*Hence from Theorem 3 of Cline (1986) or Theorem 2.1 of Liu and Tang (2010), we know  $H \in \mathcal{S}$ , which means there are two distributions  $F$  and  $G$  such that  $H \in \mathcal{S}$ , but  $F$  (and  $G$ )  $\notin \mathcal{L}(\gamma)$  for any  $\gamma \geq 0$ .*

*iii) Similarly, even if  $H \in \mathcal{L}$ , the two distributions  $F$  and  $G$  do not need to belong to the class  $\mathcal{L}(\gamma)$  for some  $\gamma \geq 0$ . Then, when  $F$  (and  $G$ )  $\notin \mathcal{L}(\gamma)$  for every  $\gamma \geq 0$ , under what conditions,  $H \in \mathcal{L}$ ? In Subsection 3.2, we will give two positive results.*

Then, from another point of view, we would like to discuss a topic on the closure property under the product distribution root. In other words, we want to know that, if  $H \in \mathcal{S}$ , under what conditions,  $F \in \mathcal{S}$ ? For this problem, which is related to an open question in Acknowledgement of Tang (2008), we also give two positive and simple results as follows.

**Theorem 1.2.** *Under one of the following two conditions, if  $H \in \mathcal{S}$ , then  $F \in \mathcal{S}$ .*

*i)  $F \in \mathcal{L}$  and*

$$\overline{H}(x) = O(\overline{F}(x/t)) \text{ for some } t \geq 1. \quad (1.2)$$

*ii)  $G$  is discontinuous and there exists a constant  $d \in D[G]$  such that*

$$\overline{H}(x) = O(\overline{F}(x/d)). \quad (1.3)$$

**Remark 1.2.** *i) There exist many distributions  $F$ ,  $G$  and  $H$  satisfying condition (1.2) or condition (1.3). For example, let  $F \in \mathcal{R}$  with a regular index  $\alpha > 0$  and  $EY^{\alpha+\varepsilon} < \infty$  for some  $\varepsilon > 0$ , then by the well-known result in Breiman (1965) we have  $\overline{H}(x) \sim EY^\alpha \overline{F}(x)$ , which means condition (1.2) for all  $t \geq 1$  and condition (1.3) for all  $d \in D(G)$  both hold.*

In particular, if the distribution  $G$  has a finite support  $[0, s]$  for some constant  $s > 0$ , then it is easy to know that (1.2) holds for all  $t \geq a$ .

ii) However, there also exist distributions  $F$ ,  $G$  and  $H$  not satisfying condition (1.2) and condition (1.3) such that,  $H$  and  $F$  both belongs to the class  $\mathcal{S}$ . For instance, see Example 3.1 in section 3, for every  $t \geq 1$ ,

$$\overline{H}(x_n)/\overline{F}(x_n/t) \geq \overline{G}(x_n)/\overline{F}(x_n/t) = t^\alpha x_n \rightarrow \infty, \text{ as } n \rightarrow \infty,$$

which implies condition (1.2) and condition (1.3) do not hold.

iii) Even so, for some distributions  $F$ ,  $G$  and  $H$ , condition (1.2) or condition (1.3) is necessary in some sense. For example, let  $F$  be a continuous distribution belonging to class  $\mathcal{L} \setminus \mathcal{S}$ , and let  $G$  be a distribution belonging to class  $\mathcal{S}$ . Then by Theorem 1.1,  $H \in \mathcal{S}$ , but condition (1.2) does not hold, clearly. And then, we have the following result with a stronger conviction.

**Proposition 1.1.** *For every continuous distribution  $F$  belonging to the class  $\mathcal{L}(\gamma)$  with some  $\gamma > 0$  or the class  $\mathcal{L} \setminus \mathcal{S}$ , there exists a discontinuous distribution  $G$  supported on  $[0, \infty)$  such that  $G \notin \mathcal{L}$  and  $H \in \mathcal{S}$ , while condition (1.2) and condition (1.3) do not hold.*

**Remark 1.3.** i) From the proof of the proposition below, we can find a construction method for the distribution  $H \in \mathcal{S}$ , where distribution  $F$  does not belong to the class  $\mathcal{S}$ .

ii) We can find the distributions belonging to class  $\mathcal{L} \setminus \mathcal{S}$  in Pitman(1979), Embrechts and Goldie(1980), Murphree (1989), Leslie(1989), Lin and Wang (2012), Wang et al. (2016), and so on. So, there are a lot of such distributions  $F$  and  $G$  enabling  $H \in \mathcal{S}$ .

The rest of this paper consists of three sections. In Section 2, we give the proofs of Theorem 1.1, Theorem 1.2 and Proposition 1.1, among which Theorem 1.1 is the most important result in present paper. In Section 3, we do some discussion on the conditions (d) and other one. Finally, as an application of Theorem 1.1, we obtain asymptotics for ruin probability of a discrete-time insurance risk model with random interest rate under some more relaxed conditions than the existing results.

## 2 Proofs of the main results

In this section, we prove Theorem 1.1, Theorem 1.2 and Proposition 1.1, respectively.

### 2.1 Proof of Theorem 1.1

The converse result follows from Theorem 1.B with  $\gamma = 0$ , since  $\mathcal{S}$  is a subset of  $\mathcal{L}$ . Also, by Theorem 1.B, condition (1.1) implies that  $H \in \mathcal{L}$ . Then the only statement is left to be proved: given that  $F \in \mathcal{S}$  and  $H \in \mathcal{L}$ , we have  $H \in \mathcal{S}$ . Following Lemma 2.3(i) in Cline and Samorodnitsky (1994), noting that  $\mathcal{L}$  and  $\mathcal{S}$  are closed under scalar multiplication, we only need to prove the result for the case  $Y \geq 1$  a.s.. By  $F \in \mathcal{S}$  and  $H \in \mathcal{L}$ , there is a function  $a(\cdot) : [0, \infty) \rightarrow (0, \infty)$  satisfying

$$a(x) \uparrow \infty, a^2(x)/x \downarrow 0, \overline{F}(x - a^2(x)) \sim \overline{F}(x) \text{ and } \overline{H}(x - a(x)) \sim \overline{H}(x).$$

For the given function  $a(x)$ , we take two positive functions  $b(\cdot)$  and  $c(\cdot)$  supported on  $[0, \infty)$  such that

$$c(x) \uparrow \infty, b(x) \uparrow \infty, c(x) = o(b(x)) \text{ and } b(x)c(x) = o(a(x)), \quad (2.1)$$

and

$$\overline{G}(a(x)) = o(\overline{G}(2c(x))) \text{ and } \overline{H}(a(x)) = o(\overline{F}(b(x))). \quad (2.2)$$

In fact, there are two functions  $b_1(\cdot)$  and  $c_1(\cdot)$  supported on  $[0, \infty)$  such that  $\overline{G}(a(x)) = o(\overline{G}(2c_1(x)))$  and  $\overline{H}(a(x)) = o(\overline{F}(b_1(x)))$ . For example, for every  $n \geq 1$ , there is a positive number  $x_n$  such that

$$\overline{H}(a(x))/\overline{F}(n) < 1/n, \text{ when } x \geq x_n.$$

Without loss of generality, we set  $x_1 < x_2 < \dots < x_n \uparrow \infty$ , and define a function

$$b_1(x) = 1 \text{ for } x \in [0, x_1) \text{ and } b_1(x) = n \text{ for } x \in [x_n, x_{n+1}), \quad n \geq 1.$$

Clearly,  $b_1(x) \uparrow \infty$  and  $\overline{H}(a(x)) = o(\overline{F}(b_1(x)))$ . Further, take a function  $b(x) = \min\{b_1(x), (a(x))^{1/2}\}$  for all  $x \geq 0$ , then  $b(\cdot)$  satisfies all the requirements.

Let  $X_i$  and  $Y_i$ ,  $i = 1, 2$ , be independent copies of  $X$  and  $Y$ . For any  $x > 0$ , we have

$$\begin{aligned} \overline{H^{*2}}(x) &= P\left(X_1Y_1 + X_2Y_2 > x, \bigcap_{i=1}^2 \{1 \leq Y_i \leq x/b(x)\}\right) + P\left(X_1Y_1 + X_2Y_2 > x, \bigcup_{i=1}^2 \{Y_i > x/b(x)\}\right) \\ &=: L_1 + L_2. \end{aligned} \quad (2.3)$$

We first estimate  $L_1$ . Clearly, we have

$$\begin{aligned} L_1 &= P(X_1Y_1 + X_2Y_2 > x, 1 \leq Y_1 \leq Y_2 \leq x/b(x)) + P(X_1Y_1 + X_2Y_2 > x, 1 \leq Y_2 < Y_1 \leq x/b(x)) \\ &=: L_{11} + L_{12}. \end{aligned} \quad (2.4)$$

For  $L_{11}$ , we have the following decomposition:

$$\begin{aligned} L_{11} &\leq P(X_1Y_1 > x - c(x)a(x), 1 \leq Y_1 \leq Y_2 \leq x/b(x)) \\ &\quad + P(X_2Y_2 > x - c(x)a(x), 1 \leq Y_1 \leq Y_2 \leq x/b(x)) \\ &\quad + P(X_1Y_1 + X_2Y_2 > x, c(x)a(x) \leq X_2Y_2 \leq x - c(x)a(x), 1 \leq Y_1 \leq Y_2 \leq x/b(x)) \\ &=: L_{111} + L_{112} + L_{113}. \end{aligned} \quad (2.5)$$

Noting that  $a^2(x)/x \downarrow 0$  implies that  $a^2(x/z) \geq a^2(x)/z$  holds for all  $z \geq 1$  and  $x > 0$ . According to (2.1), for sufficiently large  $x$ , we have

$$\begin{aligned} L_{111} &\leq \int_1^{x/b(x)} P(X_1Y_1 > x - a^2(x), 1 \leq Y_1 \leq y) G(dy) \\ &= \int_1^{x/b(x)} \int_1^y \overline{F}((x - a^2(x))/z) G(dz) G(dy) \\ &\leq \int_1^{x/b(x)} \int_1^y \overline{F}((x/z) - a^2(x/z)) G(dz) G(dy) \\ &\sim \int_1^{x/b(x)} \int_1^y \overline{F}(x/z) G(dz) G(dy) \\ &= P(X_1Y_1 > x, 1 \leq Y_1 \leq Y_2 \leq x/b(x)). \end{aligned} \quad (2.6)$$

The last but one step is due to that  $\overline{F}((x/z) - a^2(x/z)) \sim \overline{F}(x/z)$  uniformly holds for  $0 < z \leq x/b(x)$ . Using a similar approach, it follows that

$$L_{112} \lesssim P(X_2Y_2 > x, 1 \leq Y_1 \leq Y_2 \leq x/b(x)). \quad (2.7)$$

For  $L_{113}$ , we have the following decomposition

$$\begin{aligned}
L_{113} &\leq P(X_1 Y_2 + X_2 Y_2 > x, c(x)a(x) < X_2 Y_2 \leq x - c(x)a(x), c(x) < Y_1 \leq Y_2 \leq x/b(x)) \\
&+ P(X_1 Y_2 + X_2 Y_2 > x, c(x)a(x) < X_2 Y_2 \leq x - c(x)a(x), 1 \leq Y_2 \leq a(x)) \\
&+ P(X_1 Y_1 + X_2 Y_2 > x, c(x)a(x) < X_2 Y_2 \leq c(x)x/b(x), a(x) < Y_2 \leq x/b(x), 1 \leq Y_1 \leq c(x)) \\
&+ P(X_1 Y_2 + X_2 Y_2 > x, c(x)x/b(x) < X_2 Y_2 \leq x - c(x)a(x), a(x) < Y_2 \leq x/b(x), 1 \leq Y_1 \leq c(x)) \\
&=: \sum_{i=1}^4 L_{113i}.
\end{aligned} \tag{2.8}$$

Firstly, for  $L_{1131}$ , we have

$$\begin{aligned}
L_{1131} &\leq P((X_1 + X_2)Y_2 > x, c(x) < Y_2 \leq x/b(x), Y_1 > c(x)) \\
&= \overline{G}(c(x)) \int_{c(x)}^{x/b(x)} \overline{F^{*2}}(x/y) G(dy) \\
&\sim \overline{G}(c(x)) \int_{c(x)}^{x/b(x)} 2\overline{F}(x/y) G(dy) \\
&= o(\overline{H}(x)).
\end{aligned} \tag{2.9}$$

Since  $F \in \mathcal{S}$  implies that  $\overline{F^{*2}}(x/y) - (1 + F(c(x)))\overline{F}(x/y) = o(\overline{F}(x/y))$  uniformly holds for  $1 \leq y \leq a(x)$ , conditioning on  $Y_2$  and  $X_2$ , it follows that

$$\begin{aligned}
L_{1132} &= \int_1^{a(x)} \int_{c(x)a(x)/y}^{(x-c(x)a(x))/y} \overline{F}((x/y) - z) F(dz) G(dy) \\
&\leq \int_1^{a(x)} \left( \overline{F^{*2}}(x/y) - \overline{F}(x/y) - \int_0^{c(x)a(x)/y} \overline{F}((x/y) - z) F(dz) \right) G(dy) \\
&\leq \int_1^{a(x)} \left( \overline{F^{*2}}(x/y) - (1 + F(c(x)))\overline{F}(x/y) \right) G(dy) \\
&= o(\overline{H}(x)).
\end{aligned} \tag{2.10}$$

Since  $\overline{H}(x) \geq \overline{F}\left(\frac{x}{c(x)} - \frac{x}{b(x)}\right) \overline{G}\left(\frac{b(x)c(x)}{b(x)-c(x)}\right)$  and  $\frac{b(x)c(x)}{b(x)-c(x)} \leq 2c(x)$  for sufficiently large  $x$ , by (2.2), we have

$$\begin{aligned}
L_{1133} &\leq P\left(X_1 Y_1 > x - \frac{c(x)x}{b(x)}, a(x) < Y_2 \leq \frac{x}{b(x)}, 1 \leq Y_1 \leq c(x)\right) \\
&\leq \overline{F}\left(\frac{x}{c(x)} - \frac{x}{b(x)}\right) \overline{G}(a(x)) \\
&\leq \left(\overline{H}(x) \overline{F}\left(\frac{x}{c(x)} - \frac{x}{b(x)}\right) \overline{G}(a(x))\right) / \left(\overline{F}\left(\frac{x}{c(x)} - \frac{x}{b(x)}\right) \overline{G}\left(\frac{b(x)c(x)}{b(x)-c(x)}\right)\right) \\
&= o(\overline{H}(x)).
\end{aligned} \tag{2.11}$$

Now, conditioning on  $Y_2$  and  $X_2$  again, we have

$$L_{1134} \leq \int_{a(x)}^{x/b(x)} \int_{(c(x)x)/(b(x)y)}^{(x-c(x)a(x))/y} \overline{F}((x/y) - z) F(dz) G(dy)$$



$$\begin{aligned}
&\leq \int_{a(x)}^{x/b(x)} \left( \overline{F^{*2}}(x/y) - \overline{F}(x/y) - \int_0^{(c(x)x)/(b(x)y)} \overline{F}((x/y) - z) F(dz) \right) G(dy) \\
&\leq \int_{a(x)}^{x/b(x)} \left( \overline{F^{*2}}(x/y) - (1 + F(c(x))\overline{F}(x/y)) \right) G(dy) \\
&= o(\overline{H}(x)).
\end{aligned} \tag{2.12}$$

Substituting (2.9)-(2.12) into relation (2.8) yields that

$$L_{113} = o(\overline{H}(x)). \tag{2.13}$$

Substituting (2.6), (2.7) and (2.13) into relation (2.5) yields that

$$\begin{aligned}
L_{11} &\lesssim P(X_1 Y_1 > x, 1 \leq Y_1 \leq Y_2 \leq x/b(x)) + P(X_2 Y_2 > x, 1 \leq Y_1 \leq Y_2 \leq x/b(x)) \\
&\quad + o(\overline{H}(x)).
\end{aligned} \tag{2.14}$$

By symmetry, we obtain that

$$\begin{aligned}
L_{12} &\lesssim P(X_1 Y_1 > x, 1 \leq Y_2 < Y_1 \leq x/b(x)) + P(X_2 Y_2 > x, 1 \leq Y_2 < Y_1 \leq x/b(x)) \\
&\quad + o(\overline{H}(x)).
\end{aligned} \tag{2.15}$$

Clearly, it follows from (2.4), (2.14) and (2.15) that

$$L_1 \lesssim 2P(X_1 Y_1 > x, 1 \leq Y_1 \leq x/b(x)). \tag{2.16}$$

Now, we are ready to estimate  $L_2$ . Clearly, we have

$$\begin{aligned}
L_2 &\leq 2P(X_1 Y_1 + X_2 Y_2 > x, Y_1 > x/b(x)) \\
&\leq 2P(X_1 Y_1 > x - a(x), Y_1 > x/b(x)) + 2P(X_2 Y_2 > a(x), Y_1 > x/b(x)) \\
&=: 2L_{21} + 2L_{22}.
\end{aligned} \tag{2.17}$$

For  $L_{21}$ , it follows that

$$\begin{aligned}
L_{21} &= P(X_1 Y_1 > x - a(x)) - P(X_1 Y_1 > x - a(x), Y_1 \leq x/b(x)) \\
&\leq P(X_1 Y_1 > x - a(x)) - P(X_1 Y_1 > x, Y_1 \leq x/b(x)) \\
&= \overline{H}(x) + o(\overline{H}(x)) - P(X_1 Y_1 > x, Y_1 \leq x/b(x)) \\
&= P(X_1 Y_1 > x, Y_1 > x/b(x)) + o(\overline{H}(x)).
\end{aligned} \tag{2.18}$$

And for  $L_{22}$ , from (2.2), we have

$$L_{22} = \overline{H}(x)\overline{H}(a(x))\overline{G}(x/b(x))/\overline{H}(x) \leq \overline{H}(x)\overline{H}(a(x))/\overline{F}(b(x)) = o(\overline{H}(x)). \tag{2.19}$$

Hence by (2.3) and (2.16)-(2.19), it holds that

$$\overline{H^{*2}}(x) \lesssim 2\overline{H}(x),$$

which implies that  $H \in \mathcal{S}$ .



## 2.2 Proof of Theorem 1.2

i) For above  $t \geq 1$ , let  $F_1$  be a distribution such that  $\overline{F}_1(x) = \overline{F}(x/t)$  for all  $x$ . By  $H \in \mathcal{S}$ ,  $F \in \mathcal{L}$ , and (1.2), we have  $F_1 \in \mathcal{L}$  and  $\overline{F}_1(x) \approx \overline{H}(x)$ , thus  $F_1 \in \mathcal{S}$ . Therefore,  $\overline{F^{*2}}(x) = \overline{F_1^{*2}}(tx) \sim 2\overline{F}_1(tx) = 2\overline{F}(x)$ , that is  $F \in \mathcal{S}$ .

ii) From the following fact

$$\begin{aligned} \overline{H}(x-2) - \overline{H}(x) &\geq \int_{(d-(d/x), d]} \overline{F}((x-2)/y) - \overline{F}(x/y) G(dy) \\ &\geq (\overline{F}((x-2)/(d-(d/x))) - \overline{F}(x/d)) G\{d\} \\ &\geq (\overline{F}((x-1)/d) - \overline{F}(x/d)) G\{d\}, \end{aligned}$$

and by  $H \in \mathcal{S}$  and (1.3), we have

$$\overline{F}((x-1)/d) - \overline{F}(x/d) = o(\overline{H}(x)) = o(\overline{F}(x/d)),$$

which implies  $F \in \mathcal{L}$ . Then by i), we can get  $F \in \mathcal{S}$ .

## 2.3 Proof of Proposition 1.1

Let  $X$  and  $Y_0$  supported on  $[0, \infty)$  be two independent random variables with continuous distributions  $F$  and  $G_1$ , satisfying  $F \in \mathcal{L}(\gamma)$  for some  $\gamma > 0$  or  $F \in \mathcal{L} \setminus \mathcal{S}$  and  $G_1 \in \mathcal{S}$ , respectively. Denote the distribution of  $XY_0$  by  $H_1$ . Since  $G_1$  is continuous, by Theorem 1.1, we have  $H_1 \in \mathcal{S}$ .

Further, using the method in Example 3.1 of Xu et al. (2016), we can construct a new random variable  $Y$  with discontinuous distribution  $G$ , satisfying  $G \notin \mathcal{L}$  and  $\overline{G}(x) \approx \overline{G}_1(x)$ , which implies  $\overline{H}(x) \approx \overline{H}_1(x)$ . On the other hand, we can get  $H \in \mathcal{L}$  directly from Theorem 1.B. Combining with the above facts, we know that  $H \in \mathcal{S}$ .

Finally, conditions (1.2) and (1.3) are obviously unsatisfied.

## 3 Discussion

In this section, we give an example for a further discussion on the conditions in Theorem 1.A and give two sufficient conditions for  $H \in \mathcal{L}$ , respectively.

### 3.1 On the condition (d)

The following example shows that, when  $F \in \mathcal{S}$ , the condition (d) is unnecessary for  $H \in \mathcal{S}$ .

**Example 3.1.** Choose any constants  $\alpha > 0$  and  $x_1 > 4^\alpha$ . For all integers  $n \geq 1$ , let  $x_{n+1} = x_n^{1+1/\alpha}$ . Clearly,  $x_{n+1} > 4x_n$  and  $x_n \rightarrow \infty$ , as  $n \rightarrow \infty$ . We define two distributions  $F$  and  $G$  such that

$$\begin{aligned} \overline{G}(x) &= \mathbf{1}(x < 0) + (1 + (x_1^{-\alpha-1} - x_1^{-1})x) \mathbf{1}(0 \leq x < x_1) \\ &+ \sum_{n=1}^{\infty} \left( (x_n^{-\alpha} + (x_n^{-\alpha-2} - x_n^{-\alpha-1})(x - x_n)) \mathbf{1}(x_n \leq x < 2x_n) + x_n^{-\alpha-1} \mathbf{1}(2x_n \leq x < x_{n+1}) \right) \end{aligned}$$

and

$$\overline{F}(x) = \mathbf{1}(x < 1) + \mathbf{1}(x \geq 1)x^{-\alpha-1}, \text{ for all } x.$$

Because the distribution  $F \in \mathcal{R} \subset \mathcal{S}$  and is continuous, by Theorem 1.1,  $H \in \mathcal{S}$ . From

$$\overline{G}(2x_n - 1)/\overline{G}(2x_n) = 2 - x_n^{-1} \rightarrow 2, \quad n \rightarrow \infty$$

we know  $G \notin \mathcal{L}$ . Direct calculation leads to

$$\overline{H}(x) = x^{-\alpha-1} \int_0^x y^{\alpha+1} G(dy) + \overline{G}(x), \text{ for all } x.$$

Thus, when  $x \in [x_n, 2x_n)$ ,

$$\begin{aligned} \overline{H}(x) &= x^{-\alpha-1} \left( \int_0^{x_1} (x_1^{-1} - x_1^{-\alpha-1}) y^{\alpha+1} dy + \sum_{i=1}^{n-1} \int_{x_i}^{2x_i} (x_i^{-\alpha-1} - x_i^{-\alpha-2}) y^{\alpha+1} dy \right. \\ &\quad \left. + \int_{x_n}^x (x_n^{-\alpha-1} - x_n^{-\alpha-2}) y^{\alpha+1} dy \right) + \overline{G}(x); \end{aligned} \quad (3.1)$$

when  $x \in [2x_n, x_{n+1})$ ,

$$\overline{H}(x) = x^{-\alpha-1} \left( \int_0^{x_1} (x_1^{-1} - x_1^{-\alpha-1}) y^{\alpha+1} dy + \sum_{i=1}^n \int_{x_i}^{2x_i} (x_i^{-\alpha-1} - x_i^{-\alpha-2}) y^{\alpha+1} dy \right) + \overline{G}(x). \quad (3.2)$$

Now, we prove  $H \in \mathcal{D}$ . In fact, when  $x \in [x_n, 4x_n)$ , by (3.2), we have

$$\overline{H}(x/2)/\overline{H}(x) \leq \overline{H}(x_n/2)/\overline{H}(4x_n) \leq 2^{3+3\alpha} + 2^{2+2\alpha} < \infty;$$

when  $x \in [4x_n, x_{n+1})$ , by the inequality  $(a+b)/(c+d) \leq (a/b) + (c/d)$ , for all  $a, b, c, d > 0$  and (3.2), then

$$\overline{H}(x/2)/\overline{H}(x) \leq 2^{\alpha+1} + 1 < \infty.$$

From (3.1), we know that  $\overline{G}(x_n)/\overline{H}(x_n) \rightarrow 1$ , as  $n \rightarrow \infty$ . Then by  $H \in \mathcal{D}$ , we can not get  $\overline{G}(x) = o(\overline{H}(bx))$  for any  $b > 0$ . That is for any positive function  $a(x)$  satisfying  $a(x) \uparrow \infty$  and  $a(x)/x \downarrow 0$ , the condition (d) does not hold.

### 3.2 On the long-tailed distribution

The following two results show that, even if two distributions  $F$  and  $G$  do not belong to the class  $\mathcal{L}(\gamma)$  for any  $\gamma \geq 0$ , under certain conditions, the product convolution  $H$  is still long-tailed.

**Theorem 3.1.** *Let  $F$  and  $G$  be two continuous distribution. If there exists a function  $a(\cdot) : [0, \infty) \rightarrow (0, \infty)$  satisfying  $a(x) \uparrow \infty$ ,  $x/a(x) \uparrow \infty$  and*

$$\overline{G}(a(x)) = O(\overline{H}(x)), \quad \overline{F}(x/a(x)) = O(\overline{H}(x)), \quad (3.3)$$

*then  $H \in \mathcal{L}$ .*

*Proof.*

$$\begin{aligned}
\overline{H}(x-1) &= P(XY > x-1, Y \leq a(x)) + P(XY > x-1, Y > a(x)) \\
&\leq P(XY > x-1, X \geq (x-1)/a(x)) + P(XY > x-1, Y > a(x)) \\
&= \left( \int_{(x-1)/a(x)}^{(x-1)/a(x-1)} + \int_{(x-1)/a(x-1)}^{x/a(x)} + \int_{x/a(x)}^{\infty} \right) \overline{G}((x-1)/y)F(dy) + \int_{a(x)}^{\infty} \overline{F}((x-1)/y)G(dy) \\
&\leq \int_{x/a(x)}^{\infty} \overline{G}((x-1)/y)F(dy) + \int_{a(x)}^{\infty} \overline{F}((x-1)/y)G(dy) \\
&\quad + \overline{F}((x-1)/a(x))\overline{G}(a(x-1)) + \overline{F}((x-1)/a(x-1))\overline{G}((x-1)a(x)/x) \tag{3.4}
\end{aligned}$$

Clearly, it follows from (3.3) that

$$\overline{F}((x-1)/a(x))\overline{G}(a(x-1)) + \overline{F}((x-1)/a(x-1))\overline{G}((x-1)a(x)/x) = o(\overline{H}(x-1)). \tag{3.5}$$

Moreover, since  $F$  and  $G$  are continuous on  $[0, \infty)$ , then they are uniformly continuous on  $[0, \infty)$ . Hence, for any fixed  $\epsilon > 0$  and for sufficiently large  $x$ , we have that  $\overline{G}((x-1)/y) < \overline{G}(x/y) + \epsilon$  uniformly holds for  $y > x/a(x)$  and that  $\overline{F}((x-1)/y) < \overline{F}(x/y) + \epsilon$  uniformly holds for  $y > x/a(x)$ . Combining with (3.3), it follows that

$$\begin{aligned}
&\int_{x/a(x)}^{\infty} \overline{G}((x-1)/y)F(dy) + \int_{a(x)}^{\infty} \overline{F}((x-1)/y)G(dy) \\
&= \int_{x/a(x)}^{\infty} \overline{G}(x/y)F(dy) + \int_{a(x)}^{\infty} \overline{F}(x/y)G(dy) + o(\overline{H}(x)) \\
&= \overline{H}(x) + o(\overline{H}(x)). \tag{3.6}
\end{aligned}$$

Thus that  $H \in \mathcal{L}$  follows from (3.4)-(3.6).  $\square$

**Theorem 3.2.** *Let  $F$  and  $G$  be two continuous distributions. If*

$$\overline{F}(x-1/x) \sim \overline{F}(x) \text{ and } \overline{G}(x-1/x) \sim \overline{G}(x) \tag{3.7}$$

*and there exists a function  $a(\cdot) : [0, \infty) \rightarrow (0, \infty)$  satisfying  $a(x) \uparrow \infty$ ,  $x/a(x) \uparrow \infty$  and*

$$\overline{G}(a(x)) = O(\overline{H}(x)), \quad \overline{F}(a(x)) = O(\overline{H}(x)), \tag{3.8}$$

*then  $H \in \mathcal{L}$ .*

*Proof.* Without loss of generality, we assume  $a(x) > \sqrt{x}$ . Consider the following expression

$$\begin{aligned}
\overline{H}(x-1) &= P(XY > x-1, Y \leq \sqrt{x}) + P(XY > x-1, Y > \sqrt{x}) \\
&= P(XY > x-1, X > \sqrt{x}) + P(XY > x-1, Y > \sqrt{x}) \\
&\quad + P(XY > x-1, (x-1)/\sqrt{x} < X \leq \sqrt{x}) - P(X > (x-1)/\sqrt{x}, Y > \sqrt{x}) \\
&\leq \left( \int_{\sqrt{x}}^{a(x)} + \int_{a(x)}^{\infty} \right) \overline{G}((x-1)/y)F(dy) + \left( \int_{\sqrt{x}}^{a(x)} + \int_{a(x)}^{\infty} \right) \overline{F}((x-1)/y)G(dy) \\
&\quad + (\overline{F}((x-1)/\sqrt{x}) - \overline{F}(\sqrt{x}))(\overline{G}((x-1)/\sqrt{x}) - \overline{G}(\sqrt{x})) - \overline{F}(\sqrt{x})\overline{G}(\sqrt{x}). \tag{3.9}
\end{aligned}$$

From (3.7) and  $\overline{H}(x) \geq \overline{F}(\sqrt{x})\overline{G}(\sqrt{x})$ , it is easy to know

$$(\overline{F}((x-1)/\sqrt{x}) - \overline{F}(\sqrt{x}))(\overline{G}((x-1)/\sqrt{x}) - \overline{G}(\sqrt{x})) = o(\overline{H}(x)). \quad (3.10)$$

Then by (3.7), for sufficiently large  $x$ , we have

$$\begin{aligned} & \int_{\sqrt{x}}^{a(x)} \overline{G}((x-1)/y)F(dy) + \int_{\sqrt{x}}^{a(x)} \overline{F}((x-1)/y)G(dy) \\ & \leq \int_{\sqrt{x}}^{a(x)} \overline{G}\left(\frac{x}{y} - \frac{y}{x}\right)F(dy) + \int_{\sqrt{x}}^{a(x)} \overline{F}\left(\frac{x}{y} - \frac{y}{x}\right)G(dy) \\ & = \int_{\sqrt{x}}^{a(x)} \overline{G}(x/y)F(dy) + \int_{\sqrt{x}}^{a(x)} \overline{F}(x/y)G(dy) + o(\overline{H}(x)). \end{aligned} \quad (3.11)$$

Since  $F$  and  $G$  are continuous on  $[0, \infty)$ , then they are uniformly continuous on  $[0, \infty)$ . Using a similar approach as in Theorem 3.1 and combining with (3.8), it follows that

$$\begin{aligned} & \int_{a(x)}^{\infty} \overline{G}((x-1)/y)F(dy) + \int_{a(x)}^{\infty} \overline{F}((x-1)/y)G(dy) \\ & = \int_{a(x)}^{\infty} \overline{G}(x/y)F(dy) + \int_{a(x)}^{\infty} \overline{F}(x/y)G(dy) + o(\overline{H}(x)) \end{aligned} \quad (3.12)$$

Substituting (3.10)-(3.12) into relation (3.9) yields that

$$\begin{aligned} \overline{H}(x-1) &= \int_{\sqrt{x}}^{\infty} \overline{G}(x/y)F(dy) + \int_{\sqrt{x}}^{\infty} \overline{F}(x/y)G(dy) - \overline{F}(\sqrt{x})\overline{G}(\sqrt{x}) + o(\overline{H}(x)) \\ &= \overline{H}(x) + o(\overline{H}(x)), \end{aligned}$$

which implies  $H \in \mathcal{L}$ . □

**Remark 3.1.** When

$$\overline{F}(x) = \overline{G}(x) = \mathbf{1}(x \leq 0) + e^{-x^2}\mathbf{1}(x > 0) \text{ for all } x,$$

it follows from Lemma 2.1 of Arendarczyk and Dębicki (2011) that  $\overline{H}(x) \sim \sqrt{\pi x}e^{-2x}$ , which yields that  $H \in \mathcal{L}(2)$ . Notice that (3.8) holds for all  $a(x) = x^\beta$ ,  $\beta \in (1/2, 1)$  but (3.7) does not hold since  $\overline{F}(x-1/x) \sim e^2\overline{F}(x)$ . This shows the condition (3.7) in Theorem 3.2 may be necessary for  $H \in \mathcal{L}$  in certain sense.

## 4 Application to the risk theory

In this section, we consider a discrete-time insurance risk model with insurance and financial risks, which was proposed by Nyrhinen (1999, 2001).

For each  $i \geq 1$ , within period  $i$ , we denote the net insurance loss (i.e. the total claim amount minus the total premium income) by a real-valued r.v.  $Z_i$  with common distribution  $F_0$  supported on  $(-\infty, \infty)$ . Further, we denote the distribution of  $Z_1^+ = Z_1\mathbf{1}(Z_1 \geq 0) =: X$  by  $F$ . Suppose that the insurer makes both risk-free and risky investments, which lead to an overall stochastic discount factor  $Y_i$  with common distribution  $G$  supported on  $(0, a]$  for some  $0 < a < \infty$  or  $(0, \infty)$ ,

from time  $i$  to time  $i - 1$ ,  $i \geq 1$ . We assume the  $\{Z_i, i \geq 1\}$  and  $\{Y_i, i \geq 1\}$  are two sequences of independent random variables, and  $\{Z_i, i \geq 1\}$  is independent of  $\{Y_i, i \geq 1\}$ . They are called insurance risks and financial risks respectively in Tang and Tsitsiashvili (2003, 2004). For each positive integer  $n$ , the sum

$$S_n = \sum_{i=1}^n Z_i \prod_{j=1}^i Y_j \quad (4.1)$$

represents the stochastic discount value of aggregate net losses up to time  $n$ . Then the finite-time ruin probability by time  $n$  can be defined as

$$\psi(x; n) = P\left(\max_{1 \leq k \leq n} S_k > x\right) \quad (4.2)$$

where  $x \geq 0$  is interpreted as the initial wealth of the insurer. For each  $i \geq 1$ , we denote the distribution function of  $Z_i^+ \prod_{j=1}^i Y_j$  by  $H_i$ . Clearly,  $H_1 = H$ .

There are a lot of papers studying the asymptotic behavior of the ruin probability in this model. The reader can refer to Goovaerts et al. (2005), Zhang et al. (2009), Yi et al. (2011), Chen (2011), Cheng et al. (2012), Zhou et al. (2012), Yang and Wang (2013), Huang et al. (2014), Li and Tang (2015), and so on. After using the new result on product convolution, we can get a different version on the finite-time ruin probability.

**Theorem 4.1.** *Assume  $F \in \mathcal{S}$ , then  $S_n$  follows a subexponential distribution for each  $n \in \mathbb{N}$  if and only if  $D[F] = \emptyset$ , or when  $D[F] \neq \emptyset$ , it holds that*

$$\overline{G}(x/d) - \overline{G}((x+1)/d) = o(\overline{H}(x)) \quad \text{for all } d \in D[F]. \quad (4.3)$$

Further,

$$\psi(x; n) \sim P(S_n > x) \sim \sum_{i=1}^n \overline{H}_i(x) \quad (4.4)$$

if there is a function  $a(\cdot) : [0, \infty) \rightarrow (0, \infty)$  satisfying  $a(x) \uparrow \infty$ ,  $a(x)/x \downarrow 0$  and  $\overline{G}(a(x)) = o(\overline{H}(x))$ .

In order to prove the theorem, we need the following two lemmas. The first lemma is due to Embrechts and Goldie (1980) for support  $[0, \infty)$  and Tang and Tsitsiashvili (2003) for support  $(-\infty, \infty)$ .

**Lemma 4.1.** *Let  $V_1$  and  $V_2$  be two distributions on  $(-\infty, \infty)$  and let  $V = V_1 * V_2$ . If  $V_1 \in \mathcal{S}$ ,  $V_2 \in \mathcal{L}$  and  $\overline{V}_2(x) = O(\overline{V}_1(x))$ , then  $V \in \mathcal{S}$  and  $\overline{V}(x) \sim \overline{V}_1(x) + \overline{V}_2(x)$ .*

**Lemma 4.2.** *Assume that the distribution  $F_0$  (or equivalently,  $F$ ) is continuous on  $[0, \infty)$ , then the distribution  $H$  is continuous on  $(0, \infty)$ .*

*Proof.* Let  $G\{x\} = G(x) - G(x-)$ . For any  $x_0 > 0$ , since  $F_0$  is continuous on  $[0, \infty)$  and bounded by 1, for any  $x_0 > 0$ , by the dominant convergence theorem, we have

$$\lim_{x \rightarrow x_0} H(x) = G\{0\} + \lim_{x \rightarrow x_0} \int_{0+}^{\infty} F(x/y) G(dy) = G\{0\} + \int_{0+}^{\infty} \lim_{x \rightarrow x_0} F(x/y) G(dy) = H(x_0),$$

which implies the conclusion.  $\square$

*Proof of Theorem 4.1.* Clearly, the necessity part is immediately obtained from Theorem 4.1 and  $n = 1$ . We prove the sufficiency part by induction.

When  $n = 1$ , it is obvious that the distribution of  $S_1$  belongs to the class  $\mathcal{S}$ . We assume that  $S_n$  follows a subexponential distribution. We prove that  $S_{n+1}$  also follows a subexponential distribution. Notice that

$$S_{n+1} = \sum_{i=1}^{n+1} Z_i \prod_{j=1}^i Y_j \stackrel{d}{=} \sum_{i=1}^{n+1} Z_i \prod_{j=i}^{n+1} Y_j \stackrel{d}{=} Y_{n+1}(S_n + Z_{n+1}), \quad n \in \mathbb{N}. \quad (4.5)$$

If  $\overline{G}(1) = 0$ , by  $F \in \mathcal{S}$ , we have

$$P(S_n > x) \leq P\left(\sum_{i=1}^n Z_i^+ > x\right) \sim n\overline{F}(x), \quad (4.6)$$

If  $\overline{G}(1) > 0$ , then

$$P(S_n > x) \geq P\left(Z_n Y_n > x, \bigcap_{i=1}^{n-1} \{Z_i > 0\}\right) = \overline{H}(x)(\overline{F}(0))^{n-1} \geq \overline{G}(1)(\overline{F}(0))^{n-1}\overline{F}(x). \quad (4.7)$$

Thus, because  $S_n$  follows a subexponential distribution and  $F \in \mathcal{S}$ , and by Lemma 4.1, we know that  $S_n + Z_{n+1}$  (or equivalently,  $(S_n + Z_{n+1})^+$ ) also follows a subexponential distribution and

$$P(S_n + Z_{n+1} > x) \sim P(S_n > x) + P(Z_{n+1} > x). \quad (4.8)$$

Further, if  $F$  is continuous on  $(0, \infty)$ , then the distribution  $H$ , thus the distribution of  $S_n + Z_{n+1}$  is also one. Therefore, by Theorem 1.1, the distribution of  $((S_n + Z_{n+1})^+)Y_{n+1}$ , thus  $S_{n+1}$ , belongs to the class  $\mathcal{S}$ .

If  $F$  is not continuous on  $(0, \infty)$ , we denote the set of all discontinuities of  $Z_{n+1} + T_n$  by  $D_n$ . If  $D_n = \emptyset$ , the conclusion for  $S_{n+1}$  holds automatically from Theorem 1.1. When  $D_n \neq \emptyset$ , then for any  $c \in D_n$ , we take a constant  $d \in D[F]$ . If  $d \leq c$ , by (4.7),  $\overline{G}(c/d) > 0$  and  $\overline{F}(0) > 0$  we can get

$$\begin{aligned} \overline{G}(x/c) - \overline{G}((x+1)/c) &\leq P(S_{n+1} > x)(\overline{G}(x/c) - \overline{G}(x/c + 1/d))/(\overline{G}(c/d)P(S_n + Z_{n+1} > dx/c)) \\ &\leq P(S_{n+1} > x)(\overline{G}(x/c) - \overline{G}(x/c + 1/d))/(\overline{G}(c/d)\overline{F}(0)P(S_n > dx/c)) \\ &\leq P(S_{n+1} > x)(\overline{G}(x/c) - \overline{G}(x/c + 1/d))/(\overline{G}(c/d)(\overline{F}(0))^n\overline{H}(dx/c)) \\ &= o(P(S_{n+1} > x)). \end{aligned} \quad (4.9)$$

Otherwise there exists an integer  $m$  such that  $c < d \leq mc$ . Thus, from (4.7) and  $\overline{F}(0) > 0$  we know

$$\begin{aligned} &\overline{G}(x/c) - \overline{G}((x+1)/c) \\ &\leq P(S_{n+1} > x) \sum_{i=1}^m (\overline{G}(x/c + (i-1)/d) - \overline{G}(x/c + i/d))/(\overline{H}(x)(\overline{F}(0))^n) \\ &\leq P(S_{n+1} > x) \sum_{i=1}^m (\overline{G}(x/c + (i-1)/d) - \overline{G}(x/c + i/d))/(\overline{H}(dx/c + i-1)(\overline{F}(0))^n) \\ &= o(P(S_{n+1} > x)). \end{aligned}$$

Thus from Theorem 1.1, the distribution of  $S_{n+1}$  is subexponential.

Further, since for every  $n \geq 1$ ,

$$P(S_n > x) \leq \psi(x; n) \leq P\left(\sum_{i=1}^n Z_i^+ \prod_{j=1}^i Y_j > x\right), \quad (4.10)$$

which implies that we only need to prove

$$P(S_n > x) \sim \sum_{i=1}^n \overline{H}_i(x). \quad (4.11)$$

We use induction again. As (4.11) trivially holds for  $n = 1$ , we assume that (4.11) holds for  $n$ , and we will prove that (4.11) holds for  $n + 1$ . In fact, by (4.8), we have

$$\begin{aligned} P(S_{n+1} > x) &= P((S_n + Z_{n+1})Y_{n+1} > x) \\ &= P((S_n + Z_{n+1})^+ Y_{n+1} > x) \\ &= \left( \int_0^{a(x)} + \int_{a(x)}^\infty \right) P(S_n + Z_{n+1} > x/y) G(dy) \\ &\sim \int_0^{a(x)} (P(S_n > x/y) + P(Z_{n+1} > x/y)) G(dy) \\ &\sim \int_0^\infty \left( \sum_{i=1}^n \overline{H}_i(x/y) + \overline{F}(x/y) \right) G(dy) \\ &= \sum_{i=1}^{n+1} \overline{H}_i(x). \end{aligned}$$

Hence, (4.4) follows from (4.10) and (4.11).  $\square$

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